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Homogeneous Lorentzian spaces admitting a homogeneous structure of type $\mathcal{T}_1 \oplus \mathcal{T}_3$

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Abstract

We show that a homogeneous Lorentzian space admitting a homogeneous structure of type $\mathcal{T}_1 \oplus \mathcal{T}_3$ is either a locally symmetric space or a singular homogeneous plane wave.

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A theorem by Ambrose and Singer [1], generalized to arbitrary signature in [2], states that on a reductive homogeneous space, there exists a metric connection $\bar{\nabla} = \nabla - S$, with ∇ the Levi-Cività connection, that parallelizes the Riemann tensor R, and the $(1, 2)$-tensor S, i.e. $\bar{\nabla} g = \bar{\nabla} R = \bar{\nabla} S = 0$. Since a $(1, 2)$-tensor in $D \geq 3$ decomposes into three irreps of $\mathfrak{so}(D)$, one can classify the reductive homogeneous spaces by the occurrence of one of these irreps in the tensor S [3,4]. This leads to eight different classes, which range from the maximal, denoted by $\mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \mathcal{T}_3$, to the minimal $\{0\}$. Clearly, homogeneous spaces of type $\{0\}$ are just symmetric spaces. Moreover, also the homogeneous spaces admitting a homogeneous structure of type $\mathcal{T}_i$ ($i = 1, 2$ or 3) have been characterized. For the case at hand it is worth knowing that the homogeneous spaces with a $\mathcal{T}_3$ structure, for which S corresponds to a three-form, are naturally reductive spaces [3,4] and that strictly Riemannian

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homogeneous $\mathcal{T}_1$ spaces are locally symmetric spaces [3]. Since a homogeneous structure of type $\mathcal{T}_1$ is defined by an invariant vector field ξ, one must distinguish between two cases in the Lorentzian setting: the non-degenerate case, for which ξ is a space- or time-like vector, and the degenerate case, when ξ is a null vector. In the former case, Gadea and Oubiña [4] proved that, analogously to the strictly Riemannian case, the space is locally symmetric. In the degenerate case, Montesinos Amilibia [5] showed that a homogeneous Lorentzian space admitting a degenerate $\mathcal{T}_1$ structure is a time-independent singular homogeneous plane wave [6]. A small calculation shows that a generic, i.e. time-dependent, singular homogeneous plane wave admits a degenerate $\mathcal{T}_1 \oplus \mathcal{T}_3$ structure, see, e.g. Appendix A. (By a (non-)degenerate $\mathcal{T}_1 \oplus \mathcal{T}_3$ structure, we mean that the vector field ξ characterizing the $\mathcal{T}_1$ contribution has (non-)vanishing norm.) This then automatically leads to the question of whether the singular homogeneous plane waves exhaust the degenerate case in the $\mathcal{T}_1 \oplus \mathcal{T}_3$ class. As we will see, the answer is affirmative.

In the $\mathcal{T}_1 \oplus \mathcal{T}_3$ case, the homogeneous structure is given by [3]

$$\bar{\nabla}_X Y - \nabla_X Y = -S_X Y = -T_X Y - g(X, Y)\xi + \alpha(Y)X,$$

where we have defined $\alpha(X) = g(\xi, X)$ and $T_X Y(= -T_Y X)$ is the $\mathcal{T}_3$ contribution. Since the metric and S are parallel under $\bar{\nabla}$ and ξ is the contraction of S, it follows that $\bar{\nabla}\xi = 0$ or, written differently:

$$\nabla_X \xi = T_X \xi + \alpha(X)\xi - \alpha(\xi)X.$$

This equation, together with the fact that T is a three-form, implies that $\nabla_\xi \xi = 0$, i.e. ξ is a geodesic vector.

Given an isometry algebra $\mathfrak{g}$ (i.e. the Lie algebra of a Lie group acting transitively by isometries on a given homogeneous space), with a reductive split $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$, where $\mathfrak{h} \subseteq \mathfrak{so}(1, n+1)$ is the isotropy subalgebra, it is possible, and usually done, to identify $\mathfrak{m}$ with $\mathbb{R}^{1,n+1}$; the action of $\mathfrak{h}$ on $\mathfrak{m}$ can then be given by the vector representation of $\mathfrak{so}(1, n+1)$ [7]. This identification enables one to express the algebra in terms of S and the curvature $\bar{R}$ as, limiting ourselves to the $\mathfrak{m} \times \mathfrak{m}$ commutator,

$$[X, Y] = S_X Y - S_Y X + \bar{R}(X, Y), \tag{1}$$

where S and $\bar{R}$ are evaluated at some point p. In the above formula, $\bar{R}$ signals the presence of $\mathfrak{h}$ in $[\mathfrak{m}, \mathfrak{m}]$. From now on, we only consider this Lie algebra and all the relevant tensor fields are evaluated at a specific point, even though this is not stated explicitly.

Up to this point not too much has been said about $\mathfrak{h}$, and in fact not too much can be said. It is known, however [7], that a tensor field parallelized by $\bar{\nabla}$, when evaluated at a point corresponds to an $\mathfrak{h}$-invariant tensor. Since in this article we take ξ (an $\mathfrak{h}$-invariant vector field as $\bar{\nabla}\xi = 0$) to be non-vanishing, this means that $\mathfrak{h} \subseteq \mathfrak{so}(n+1)$ when ξ is light-like, $\mathfrak{h} \subseteq \mathfrak{so}(1, n)$ when ξ is space-like and $\mathfrak{h} \subseteq \mathfrak{iso}(n)$ when ξ is null.

Let us briefly outline the manner in which we arrive at our results: given a reductive homogeneous space with reductive split $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$, the subalgebra $\mathfrak{g}' = \mathfrak{m} + [\mathfrak{m}, \mathfrak{m}] = \mathfrak{m} + \mathfrak{h}'$ is an ideal of $\mathfrak{g}$. It is this ideal, which is the Lie algebra of a Lie group still acting transitively that we will consider; we will say that an element of $\mathfrak{h}$ appears in the algebra if it is an element of $\mathfrak{h}'$. Given the homogeneous structure, we can then, following Eq. (1), write down the maximal form of the algebra compatible with the homogeneous structure. Since we are dealing with a Lie algebra, we can then use the Jacobi identities to constrain the structure constants; after a redefinition of some generators in $\mathfrak{m}$, corresponding to the choice of a different reductive split, this leads to a recognizable result. Since the non-degenerate case is far less involved than the degenerate case, and gives a better idea of the manipulations used, it will be discussed before the degenerate case.

1. The non-degenerate case

Let $\mathfrak{m}$ be spanned by the generators V and Z_i ($i = 1, \ldots, n$), which in this case we can take to satisfy

$$\begin{aligned}
&\langle V, V\rangle = \aleph, \qquad \alpha(V) = \lambda = \aleph|\lambda|,\\
&\langle Z_i, Z_j\rangle = \eta_{ij},\ \alpha(Z_i) = 0,\\
&\langle V, Z_i\rangle = 0,
\end{aligned}$$

where $\aleph = \pm 1$ distinguishes between the time-like (for $\aleph = -1$) and the space-like (for $\aleph = 1$) cases and $\eta = \mathrm{diag}(-\aleph, 1, \ldots, 1)$. As is mentioned in the introduction, $\mathfrak{h}$ is contained in either $\mathfrak{so}(n+1)$ (for $\aleph = -1$) or $\mathfrak{so}(1, n)$ (for $\aleph = 1$) and the relevant non-vanishing commutation relations are

$$\begin{aligned}
&[M_{ij}, M_{kl}] = \eta_{jk}\, M_{il} - \eta_{ik}\, M_{jl} + \eta_{jl}\, M_{ki} - \eta_{il}\, M_{kj},\\
&[M_{ij}, Z_k] = \eta_{jk}\, Z_i - \eta_{ik}\, Z_j.
\end{aligned}$$

Once again, let us stress that not every M needs appear, but the elements of $\mathfrak{h}'$ can be written as combinations of the M's, and their commutation relations are induced by the ones above.

With respect to the chosen basis we can decompose $2T_V Z_i = F_i^{\,j} Z_j$ and $2T_{Z_i} Z_j = \aleph F_{ij} V + C_{ij}^k Z_k$, which allows us to write

$$[V, Z_i] = \lambda\, Z_i + F_i^{\,j} Z_j + \overline{R}(V, Z_i), \quad [Z_i, Z_j] = \aleph F_{ij} V + C_{ij}^k Z_k + \overline{R}(Z_i, Z_j).$$

Let us then, following the strategy outlined above, check the Jacobi identities. The first one is the (V, Z_i, Z_j) identity, which leads to $F = 0$ and

$$\frac{\lambda}{2} C_{ijk} = R_{jik} - R_{ijk}, \tag{2}$$

$$2\lambda\, S_{ij}^{mn} = C_{ij}^k\, R_k^{mn}, \tag{3}$$

where we expanded $\overline{R}(V, Z_i) = R_i^{mn}\, M_{mn}$ and $\overline{R}(Z_i, Z_j) = S_{ij}^{mn}\, M_{mn}$. Since $F = 0$ we can redefine

$$Y_i = Z_i + \lambda^{-1}\, R_i^{mn} M_{mn},$$

from which we trivially find

$$[V, Y_i] = \lambda\, Y_i,$$

which at once implies that $C = 0$, by Eq. (2), and also that $S = 0$ thanks to Eq. (3). So the, quite remarkable, result is that a Lorentzian homogeneous space admitting a non-degenerate homogeneous structure of type $\mathcal{T}_1 \oplus \mathcal{T}_3$, also admits a non-degenerate $\mathcal{T}_1$ structure. Combining this with the results of Gadea and Oubiña [4], we have proved the following result.

Proposition 1. *A connected homogeneous Lorentzian space admitting a non-degenerate $\mathcal{T}_1 \oplus \mathcal{T}_3$ structure is a locally symmetric space.*

2. The degenerate case

In the degenerate case, we can choose the generators U, V and Z_i ($i = 1, \ldots, n$) spanning $\mathfrak{m}$ such that $\alpha(U) = \lambda \neq 0$, $\alpha(V) = \alpha(Z_i) = 0$. The invariant norm is then $\langle U, V\rangle = 1$ and $\langle Z_i, Z_j\rangle = \delta_{ij}$ and we decompose the $\mathcal{T}_3$ contribution to S as

$$2T(U, V, Z_i) = W_i, \qquad 2T(U, Z_i, Z_j) = F_{ij},$$
$$2T(Z_i, Z_j, Z_k) = C_{ijk}, \quad 2T(V, Z_i, Z_j) = \aleph_{ij},$$

where F, $\aleph$ and C are totally antisymmetric. Given these abbreviations we can write the most general $\mathfrak{m} \times \mathfrak{m}$ commutators as

$$[U, V] = \lambda\, V + W^i\, Z_i + \overline{R}(U, V), \quad [U, Z_i] = \lambda\, Z_i + F_i^{\ j}\, Z_j - W_i\, U + \overline{R}(U, Z_i),$$
$$[V, Z_i] = W^i\, V + \aleph_i^{\ j}\, Z_j + \overline{R}(V, Z_i),$$
$$[Z_i, Z_j] = \aleph_{ij}\, U + F_{ij}\, V + C_{ijk}\, Z^k + \overline{R}(Z_i, Z_j),$$

where the various $\overline{R}$ need to be expanded in terms of the generators of $\mathfrak{h}$. Since ξ is null, we see that $\mathfrak{h} \subseteq \mathfrak{iso}(n)$, which we take to be spanned by $\overline{Z}_i$ and M_{ij} with commutation relations

$$[M_{ij}, M_{kl}] = \delta_{jk} M_{il} - \delta_{ik} M_{jl} + \delta_{jl} M_{ki} - \delta_{il} M_{kj}, \quad [M_{ij}, \overline{Z}_k] = \delta_{jk} \overline{Z}_i - \delta_{ik} \overline{Z}_j,$$
$$[M_{ij}, Z_k] = \delta_{jk} Z_i - \delta_{ik} Z_j, \quad [U, \overline{Z}_i] = Z_i, \quad [Z_i, \overline{Z}_j] = -\delta_{ij} V,$$

where it should be kept in mind that not all elements of $\mathfrak{iso}(n)$ need appear.

We can then once again start to recover the information contained in the Jacobi identities: the (U, V, Z) Jacobi identity reads

$$\begin{aligned}0 &= -2\lambda W_i\, V - \{\lambda\aleph_{ij} + F_i^k\aleph_{kj} + F_j^k\aleph_{ik} + W^k C_{kij}\}\, Z^k - [\bar{R}(U, V), Z_i] \\ &\quad - [\bar{R}(V, Z_i), U] + \aleph_i^j\,\bar{R}(U, Z_i) - 2\lambda\,\bar{R}(V, Z_i) - F_i^j\,\bar{R}(V, Z_i) + W^j\,\bar{R}(Z_i, Z_j).\end{aligned} \tag{4}$$

Cancellation of the V contribution then means that $\bar{R}(U, V) = -2\lambda W^i\bar{Z}_i + Y^{ij}M_{ij}$, which at once means that W can only be non-zero for those directions for which a $\bar{Z}$ appears. Specifically, should none appear, then $W = 0$. Let us then split the index i into some indices a and I, such that the $\bar{Z}_a$ do appear whereas the $\bar{Z}_I$ do not.

Having made the split, we can investigate the implication of having the null-boosts in the algebra. Let us start by looking at the $(U, Z_i, \bar{Z}_a)$ Jacobi: a small calculation then shows that this implies

$$\begin{aligned}0 &= -\aleph_{ia}\, U - \delta_{ia} W^i\, Z_i + W_i Z_a + C_{aik}\, Z^k - [\bar{R}(U, Z_i), \bar{Z}_a] \\ &\quad - \delta_{ia}\bar{R}(U, V) - \bar{R}(Z_i, Z_a).\end{aligned}$$

In order for the above to be true we must have that $\aleph_{ai} = C_{aij} = 0$ and that W can be non-zero only if no or only one $\bar{Z}$ appears in $\mathfrak{h}$. As was said above, the no-case already implies that $W = 0$, so we had better have a look at the case of one appearing null boost. For this we are helped by the $\mathfrak{h}$-part of the above equation. Clearly in the case when we are dealing with only one $\bar{Z}$, this amounts to the statement that $[\bar{R}(U, Z_a), \bar{Z}_a] = -\bar{R}(U, V)$, which, since there is no rotation in $\mathfrak{so}(n)$ that can take Z_a to Z_a, means that $\bar{R}(U, V) = 0$, and hence that $W_a = 0$. This then means that in all cases we have $W = 0$.

Continuing with the analysis, one can see that the $(Z_i, Z_j, \bar{Z}_a)$ Jacobi leads to

$$\aleph_{ij}\, Z_a = \delta_{ja}\aleph_i^k Z_k - \delta_{ia}\aleph_j^k Z_k, \quad [\bar{R}(Z_i, Z_j), \bar{Z}_a] = \delta_{ja}\bar{R}(U, Z_i) - \delta_{ia}\bar{R}(U, Z_j).$$

Then, using the fact that $\aleph_{ia} = 0$, one then sees that $\aleph_{IJ} = 0$ and that hence $\aleph_{ij} = 0$ when $\mathfrak{h}$ includes some null boost. In the case when there is no $\bar{Z}$, the relevant information can be obtained by picking out the V component in the (V, Z_i, Z_j) Jacobi: this implies that $\lambda\aleph_{ij} = F_i^k\aleph_{kj} + F_j^k\aleph_{ik}$, which after contraction leads to $\lambda\aleph_{ij}\aleph^{ij} = 0$ and thus implies that $\aleph = 0$.

The $\mathfrak{h}$-part of Eq. (4) then implies that $2\lambda\bar{R}(V, Z_i) = -F_i^j\bar{R}(V, Z_j)$, so that $\bar{R}(V, Z_i) = 0$. In order to then identically satisfy Eq. (4) we must have $[\bar{R}(U, V), Z_i] = 0$, so that $\bar{R}(U, V) = 0$.

Summarizing the results obtained thus far, we find that the non-trivial $\mathfrak{m}\times\mathfrak{m}$-commutators, scaling U in such a way that $\lambda = 1$ and decomposing the various $\bar{R}$'s, are

$$\begin{aligned}&[U, V] = V, \quad [U, Z_i] = (F+\delta)_{ij}Z_j + h_{ij}\bar{Z}_j + \tfrac{1}{2}R_{ijk}M_{jk}, \\ &[Z_i, Z_j] = F_{ij}V + C_{ijk}Z_k + S_{ijk}\bar{Z}_k + N_{ijkl}M_{kl}.\end{aligned}$$

Let us then continue our analysis of the Jacobi identities: the (U, Z_i, Z_j) Jacobi implies

$$h_{ij} = A_{(ij)} - \tfrac{1}{2}F_{ij}, \quad C_{ijk}h_{kl} = (F+\delta)_{ik}S_{kjl} + (F+\delta)_{jk}S_{ikl},$$

$$\tfrac{1}{2}C_{ijk}R_{kmn} = (F+\delta)_{ik}N_{kjmn} + (F+\delta)_{jk}N_{ikmn}, \tag{5}$$

$$S_{ijk} + R_{ijk} - R_{jik} = \delta_F C_{ijk} + C_{ijk}, \tag{6}$$

where we defined

$$\delta_F C_{ijk} = F_{il}C_{ljk} + F_{jl}C_{ilk} + F_{kl}C_{ijl}.$$

From Eq. (6) one sees that S must be totally antisymmetric. Denoting by $\mathfrak{S}_{(ijk)}$ the sum over the permutations (ijk), (jki) and (kij), the (Z_i, Z_j, Z_k) Jacobi results in

$$\begin{aligned}
0 &= \mathfrak{S}_{(ijk)}\, C_{jkl}S_{ilm},\\
0 &= \mathfrak{S}_{(ijk)}\, C_{jkl}N_{ilmn},\\
0 &= \mathfrak{S}_{(ijk)}[C_{jkl}C_{ilm} + 2N_{jkim}],
\end{aligned}$$

and also, since S is totally antisymmetric,

$$3S = \delta_F C. \tag{7}$$

Of course, if a $\bar{Z}_a$ occurs in $[\mathfrak{m}, \mathfrak{m}]$, then the $(U, Z_i, \bar{Z}_a)$ Jacobi implies that

$$C_{iaj} = 0,$$

$$S_{iaj} = R_{iaj}, \tag{8}$$

$$N_{iakl} = 0. \tag{9}$$

Let us then, as before, split the indices i into (a, I), where the $\bar{Z}_a$'s occur but the $\bar{Z}_I$'s do not. This means by assumption that $h_{iI} = 0$, which implies $2A_{aI} = F_{aI}$, $A_{IJ} = 0 = F_{IJ}$ and $S_{ijI} = 0$, which implies that only S_{abc} is non-zero. Furthermore, we then see that only C_{IJK} is non-vanishing. Together with Eq. (7), this then implies that $S = 0$, and we get the extra constraint

$$F_{aI}C_{IJK} = 0. \tag{10}$$

This last constraint also follows from the $(Z_i, Z_j, \bar{Z}_a)$ Jacobi, which also tells us that $N_{ijal} = 0$.

Eq. (8) then implies that only R_{IJK} and R_{aJK} are non-vanishing, and from Eq. (9) we find that only N_{IJmn} can be non-zero. We can calculate R_{aJK} from Eq. (6), which then gives $R_{aIJ} = F_{aK}C_{KIJ} = 0$ because of Eq. (10). The same equation then states $R_{IJK} - R_{JIK} = C_{IJK}$, which by means of Eq. (5) then also implies that only the N_{IJKL} can be non-vanishing.

Let us define the generator

$$Y_I = Z_I - F_{Ia}\bar{Z}_a,$$

from which we can then derive that the algebra takes on the form

$$[U, Z_a] = (F+\delta)_{ab}Z_b + (A_{ab} - \tfrac{1}{2}F_{ab})\bar{Z}_b, \quad [Z_a, Z_b] = F_{ab}V,$$

$$[U, Y_I] = Y_I + \tfrac{1}{2} R_{IJK} M_{JK}, \quad [Y_I, Y_J] = C_{IJK} Y_K + N_{IJKL} M_{KL},$$

so that the a- and the I-sectors decouple from each other.

Restricting ourselves to the I-sector and further defining

$$W_I = Y_I + \tfrac{1}{2} R_{IJK} M_{JK},$$

we immediately find $[U, W_I] = W_I$; calculating the remaining commutator, we find

$$[W_I, W_J] = (C_{IJK} - R_{IJK} + R_{JIK}) Y_K + \cdots,$$

where the $\ldots$ stands for terms in M_{JK}. Using now Eq. (6), we see that this redefinition trivializes C and by way of Eq. (5), also N.

At this point, the only difference between the algebra we deduced and the generic singular homogeneous plane wave algebra in Eq. (A.1) are the null boosts in the I-sector, that is a generator one would call $\overline{W}_I$. It is, however, always possible to extend our algebra to an algebra that does contain them; in fact this follows immediately from the consistency of the singular homogeneous plane wave algebra. Putting everything together, one sees that we obtain the isometry algebra of a generic singular homogeneous plane wave in Eq. (A.1) by, basically, choosing a different reductive split of the same algebra. Thus, we have proved the next theorem.

Theorem 2. *The underlying geometry of a connected homogeneous Lorentzian space that admits a degenerate $\mathcal{T}_1 \oplus \mathcal{T}_3$ structure is that of a singular homogeneous plane wave.*

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Appendix A. Singular homogeneous plane waves

A global coordinate system for the singular homogeneous plane waves is defined by the data[1]

$$\begin{aligned} e^+ &= \mathrm{d}z, \\ e^- &= \mathrm{d}s + [\vec{x}^T e^{zF} H e^{-zF} \vec{x} + s]\mathrm{d}z, \\ e^i &= \mathrm{d}x^i, \end{aligned}$$

[1] This form of the metric is related to the one in [6, Eq. (2.51)] by the transformations $x^+ = e^{-z}$, $x^- = -e^z s$, $\vec{z} = \vec{x}$, $A_0 = 2H$ and $f = -F$.

where the metric is defined by $\eta_{+-} = 1$ and $\eta_{ij} = \delta_{ij}$. This class of metrics admits a homogeneous structure given by the components

$$S_{++-} = -1, \quad S_{+ij} = F_{ij}, \quad S_{i+j} = -\delta_{ij} - F_{ij},$$

which corresponds to a degenerate $\mathcal{T}_1 \oplus \mathcal{T}_3$ structure.

The isometry algebra, apart from possible rotations that appear as automorphisms of the algebra, can be found to be [6]

$$\begin{aligned} &[U, V] = V, && [\bar{X}_i, \bar{X}_j] = 0 \\ &[X_i, X_j] = 2F_{ij}\, V, && [X_i, \bar{X}_j] = -\delta_{ij}\, V \\ &[U, \bar{X}_i] = X_i, && [U, X_i] = [2H - F]_{ij}\bar{X}_j + [\delta + 2F]_{ij} X_j. \end{aligned} \tag{A.1}$$

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